Quantum mechanics of many particles defined on twisted N-enlarged Newton-Hooke space-times

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Abstract

We provide the quantum mechanics of many particles moving in twisted N-enlarged Newton-Hooke space-time. In particular, we consider the example of such noncommutative system - the set of M particles moving in Coulomb field of external point-like source and interacting each other also by Coulomb potential.

1 Introduction

The idea to use noncommutative coordinates is quite old - it goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, however, there were found new formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. On the other side, the main reason for such considerations follows from many phenomenological considerations, which state that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [6], [7].

It is well-known that a proper modification of the Poincare and Galilei Hopf algebras can be realized in the framework of Quantum Groups [8], [9]. Hence, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [10], [11]), one can distinguish three types of quantum spaces [10], [11] (for details see also [12]):

1) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [13]-[15]

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu} \;, \tag{1}$$

2) Lie-algebraic modification of classical space-time [16]-[20]

$$[\, x_\mu, x_\nu \,] = i\theta^{\rho}_{\mu\nu} x_\rho \;, \tag{2}$$

and

3) Quadratic deformation of Minkowski and Galilei spaces [16], [17], [20]-[22]

$$[\, x_\mu, x_\nu \,] = i\theta^{\rho\tau}_{\mu\nu} x_\rho x_\tau \;, \tag{3}$$

with coefficients $\theta_{\mu\nu}$, $\theta^{\rho}_{\mu\nu}$ and $\theta^{\rho\tau}_{\mu\nu}$ being constants.

Besides, it has been demonstrated in [12], that in the case of so-called N-enlarged Newton-Hooke Hopf algebras $\mathcal{U}^{(N)}_0(NH_\pm)$ the twist deformation provides the new space-time noncommutativity of the form[1],[2]

$$\textbf{4)} \qquad [\, t, x_i \,] = 0 \quad , \quad [\, x_i, x_j \,] = if_\pm\left(\frac{t}{\tau}\right)\theta_{ij}(x) \;, \tag{4}$$

with time-dependent functions

$$f_+\left(\frac{t}{\tau}\right) = f\left(\sinh\left(\frac{t}{\tau}\right), \cosh\left(\frac{t}{\tau}\right)\right) \quad , \quad f_-\left(\frac{t}{\tau}\right) = f\left(\sin\left(\frac{t}{\tau}\right), \cos\left(\frac{t}{\tau}\right)\right) \;,$$

[1] $x_0 = ct$.

[2] The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [13], [14]), for quantum N-enlarged Newton-Hooke Hopf algebras.

$\theta_{ij}(x) \sim \theta_{ij} = \text{const}$ or $\theta_{ij}(x) \sim \theta_{ij}^k x_k$ and τ denoting the time scale parameter - the cosmological constant. It should be also noted that different relations between all mentioned above quantum spaces (**1)**, **2)**, **3)** and **4)**) have been summarized in paper [12].

Recently, there appeared a lot of papers dealing with classical ([23]-[29]) and quantum ([30]-[34]) mechanics, Doubly Special Relativity frameworks ([35], [36]), statistical physics ([37], [38]) and field theoretical models (see e.g. [39]), defined on quantum space-times (1), (2)[3]. Particulary, there was investigated the impact of the mentioned above deformations on dynamics of basic classical and quantum systems. Consequently, in papers [25], [26], the authors considered classical particle moving in central gravitational field defined on canonically deformed space-time (1). They demonstrated, that in such a case there is generated Coriolis force acting additionally on the moving particle. Besides, in articles [33], [25] and [34] there was analyzed classical and quantum oscillator model formulated on canonically and Lie-algebraically deformed space-time respectively. Particulary, there has been found its deformed energy spectrum as well as the corresponding equation of motion. Interesting results have been also obtained in two papers [30] and [31] concerning the hydrogen atom model defined on quantum space (1). Besides, it should be noted that there appeared article [28], which provides the link between Pioneer anomaly phenomena [42] and classical mechanics defined on κ-Galilei quantum space. Precisely, there has been demonstrated that additional force term acting on moving satellite can be identified with the force generated by space-time noncommutativity. The value of deformation parameter κ has been fixed by comparison of obtained theoretical results with observational data.

Unfortunately, in all mentioned above articles there were analyzed only the one-particle classical and quantum dynamics in the field of forces. Here, we extend such a kind of investigations to the quantum mechanics of many particles, which move in the modified twist-deformed N-enlarged Newton-Hooke space-time

$$[\,t, x_{iA}\,] = 0 \quad , \quad [\,x_{iA}, x_{jB}\,] = if(t) = if_{\pm}\left(\frac{t}{\tau}\right)\theta_{ij} \quad ; \quad i,j = 1,2,3\,, \tag{5}$$

with indices $A, B = 1, 2, \ldots, M$ labeling the particle. Further, we indicate that as in the case of one-particle quantum system there appeared additional dynamical terms generated by space-time noncommutativity. Of course, in the case of Coulomb potential for $M = 1$ and $f(t) = \theta_{ij}$ our results become the same as the ones obtained in [30] and [31] respectively.

The motivations for present studies are manyfold. First of all we extend in natural way the results for quantum one-particle model to the much more complicated many-particle system. Secondly, such investigations permit to analyze the deformations of wide class of physical models such as, for example, the noncommutative many-electron atoms or noncommutative many-atomic molecules [43], [44]. Finally, it gives a starting point for the construction of Dirac quantum mechanics for many particles defined on the relativistic counterpart of modified space-time (5).

The paper is organized as follows. In Sect. 2 we recall basic facts concerning the twisted N-enlarged Newton-Hooke space-times provided in article [12]. The third section

[3]For earlier studies see [40] and [41].

is devoted to the short review of quantum mechanics of many particles moving in commutative (classical) space. In Sect. 4 we construct the quantum many-particle model defined on modified N-enlarged Newton-Hooke space-time (5). The final remarks are presented in the last section.

2 Twisted N-enlarged Newton-Hooke space-times

In this section we turn to the twisted N-enlarged Newton-Hooke space-times equipped with two spatial directions commuting to classical time, i.e. we consider spaces of the form [12]

$$[\, t, \hat{x}_i \,] = [\, \hat{x}_1, \hat{x}_3 \,] = [\, \hat{x}_2, \hat{x}_3 \,] = 0 \quad , \quad [\, \hat{x}_1, \hat{x}_2 \,] = if(t) \;\; ; \;\; i = 1,2,3 \, . \tag{6}$$

As it was already mentioned in Introduction such a kind of quantum spaces provides the most general deformation of nonrelativistic systems. It should be noted, however, that this type of noncommutativity has been constructed explicitly only in the case of 6-enlarged Newton-Hooke Hopf algebra, with

$$\begin{aligned}
f(t) &= f_{\kappa_1}(t) = f_{\pm,\kappa_1}\left(\frac{t}{\tau}\right) = \kappa_1\, C_{\pm}^2\left(\frac{t}{\tau}\right) \,, \\
f(t) &= f_{\kappa_2}(t) = f_{\pm,\kappa_2}\left(\frac{t}{\tau}\right) = \kappa_2 \tau\, C_{\pm}\left(\frac{t}{\tau}\right) S_{\pm}\left(\frac{t}{\tau}\right) \,, \\
&\;\;\vdots \\
f(t) &= f_{\kappa_{35}}\left(\frac{t}{\tau}\right) = 86400\kappa_{35}\,\tau^{11}\left(\pm C_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1\right) \times \\
&\qquad \times \left(S_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{6}\left(\frac{t}{\tau}\right)^3 - \frac{t}{\tau}\right) \,, \\
f(t) &= f_{\kappa_{36}}\left(\frac{t}{\tau}\right) = 518400\kappa_{36}\,\tau^{12}\left(\pm C_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1\right)^2 \,,
\end{aligned} \tag{7}$$

and

$$C_{+/-}\left(\frac{t}{\tau}\right) = \cosh / \cos\left(\frac{t}{\tau}\right) \quad \text{and} \quad S_{+/-}\left(\frac{t}{\tau}\right) = \sinh / \sin\left(\frac{t}{\tau}\right) \, .$$

Besides, one can easily check that in τ approaching infinity limit the above quantum spaces reproduce the canonical (1), Lie-algebraic (2) and quadratic (3) type of space-time

noncommutativity, i.e. for $\tau \to \infty$ we get

$$\begin{aligned} f_{\kappa_1}(t) &= \kappa_1 \ , \\ f_{\kappa_2}(t) &= \kappa_2 \, t \ , \\ &\cdot \\ &\cdot \\ &\cdot \\ f_{\kappa_{35}}(t) &= \kappa_{35} \, t^{11} \ , \\ f_{\kappa_{36}}(t) &= \kappa_{36} \, t^{12} \ . \end{aligned} \tag{8}$$

Of course, for all parameters κ_a ($a = 1, ..., 36$) running to zero the above deformations disappear.

Finally, let us notice that the spaces (6) can be extended to the case of multiparticle systems as follows

$$[\, t, \hat{x}_{iA} \,] = [\, \hat{x}_{1A}, \hat{x}_{3B} \,] = [\, \hat{x}_{2A}, \hat{x}_{3B} \,] = 0 \quad , \quad [\, \hat{x}_{1A}, \hat{x}_{2B} \,] = i f(t) \delta_{AB} \ \ ; \ \ i = 1, 2, 3 \ , \tag{9}$$

with $A, B = 1, 2, \ldots, M$. It should be also observed that such an extension (blind in A, B indecies) is compatible with canonical deformation (1). Precisely, in τ approaching infinity limit the space (9) with function $f(t) = f_{\pm,\kappa_1}\left(\frac{t}{\tau}\right) = \kappa_1 \, C_{\pm}^2\left(\frac{t}{\tau}\right)$ passes into the well-known multiparticle canonical space-time proposed in [45][4] (see also [46])

$$[\, t, \hat{x}_{iA} \,] = [\, \hat{x}_{1A}, \hat{x}_{3B} \,] = [\, \hat{x}_{2A}, \hat{x}_{3B} \,] = 0 \quad , \quad [\, \hat{x}_{1A}, \hat{x}_{2B} \,] = i\kappa_1 \delta_{AB} \ . \tag{10}$$

3 Quantum mechanics of many particles moving in commutative space-time - short review

In this section we recall basic facts concerning the many-particle quantum mechanics defined on commutative space. First of all, we start with the following hamiltonian function for M interacting particles

$$H(\bar{p}_1, \ldots, \bar{p}_M; \bar{r}_1, \ldots, \bar{r}_M) = \sum_{A=1}^{M} \left(\frac{\bar{p}_A^2}{2m_A} + V_A(\bar{r}_A) \right) + \frac{1}{2} \sum_{A \neq B} V_{AB}(\bar{r}_A, \bar{r}_B) \ , \tag{11}$$

[4] It should be noted that modification of the relation (10) (blind in A, B indieces as well) is in accordance with the formal arguments proposed in [45]. Precisely, the relations (10) are constructed with adopt so-called braided tensor algebra procedure, dictated by structure of quantum R-matrix for canonical deformation [8], [9]. We would like to mention, however, that in [45] an erroneous conclusion has been stated that based on such a twisted symmetry the noncommutative quantum field theory (QFT) on the quantum space satisfying the relation in (1), and the usual commutative QFT are identical. This conclusion has been reached by a misuse of the proper transformation properties of the fields in the corresponding noncommutative space time [47].

where $\bar{p}_A = [\ x_{1A}, p_{2A}, p_{3A}\ ]$ and $\bar{r}_A = [\ x_{1A}, x_{2A}, x_{3A}\ ]$ denote the positions and momenta operators such that

$$[\ x_{iA}, x_{jB}\ ] = 0 = [\ p_{iA}, p_{jB}\ ] \quad , \quad [\ x_{iA}, p_{jB}\ ] = i\hbar\delta_{ij}\delta_{AB}\ . \tag{12}$$

Besides, present in the above formula symbol $V_A(\bar{r}_A)$ denotes the single-particle stationary potential while $V_{AB}(\bar{r}_A, \bar{r}_B)$ describes the correlations of particles. Hence, the corresponding Schroedinger equation in so-called position representation looks as follows[5]

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(\bar{r}_1, \ldots, \bar{r}_M, t) &= \left[\ \sum_{A=1}^{M}\left(\frac{1}{2m_A}\Delta_A + V_A(\bar{r}_A)\right) + \frac{1}{2}\sum_{A\neq B} V_{AB}(\bar{r}_A, \bar{r}_B)\ \right] \times \\ &\times \quad \psi(\bar{r}_1, \ldots, \bar{r}_M, t)\ , \end{aligned} \tag{13}$$

and, if one neglects the potential functions $V_{AB}(\bar{r}_A, \bar{r}_B)$ then, it takes the form

$$i\frac{\partial}{\partial t}\psi(\bar{r}_1, \ldots, \bar{r}_M, t) = \left[\ \sum_{A=1}^{M}\left(\frac{1}{2m_A}\Delta_A + V_A(\bar{r}_A)\right)\ \right]\psi(\bar{r}_1, \ldots, \bar{r}_M, t)\ . \tag{14}$$

Moreover, it is easy to see that the solution of equation (14) is given by

$$\psi(\bar{r}_1, \ldots, \bar{r}_M, t) = \psi_1(\bar{r}_1, t)\cdots\psi_M(\bar{r}_M, t)\ , \tag{15}$$

with wave functions $\psi_A(\bar{r}, t)$ satisfying the standard (one-particle) differential equation

$$i\frac{\partial}{\partial t}\psi_A(\bar{r}, t) = \left(\frac{1}{2m_A}\Delta + V_A(\bar{r})\right)\psi_A(\bar{r}, t)\ . \tag{16}$$

Usually, the potentials $V_A(\bar{r}_A)$ and $V_{AB}(\bar{r}_A, \bar{r}_B)$ remain spherically symmetric, i.e. they depend on the length of vector $\bar{r}$ and the relative positions of particles respectively

$$V_A(\bar{r}_A) = V_A(|\bar{r}_A|) \quad , \quad V_{AB}(\bar{r}_A, \bar{r}_B) = V_{AB}(|\bar{r}_A - \bar{r}_B|)\ . \tag{17}$$

Such a situation appears (for example) in the case of M electrons moving in the Coulomb field of single nucleon with charge Ze and interacting each other also by means Coulomb potential; then, we have

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(\bar{r}_1, \ldots, \bar{r}_M, t) &= \left[\ \sum_{A=1}^{M}\left(\frac{1}{2m_A}\Delta_A - \frac{Ze^2}{|\bar{r}_A|}\right) + \frac{1}{2}\sum_{A\neq B}\frac{e^2}{|\bar{r}_A - \bar{r}_B|}\ \right] \times \\ &\times \quad \psi(\bar{r}_1, \ldots, \bar{r}_M, t)\ . \end{aligned} \tag{18}$$

Finally, it should be noted that the function

$$\rho(\bar{r}_1, \ldots, \bar{r}_M, t) = |\psi(\bar{r}_1, \ldots, \bar{r}_M, t)|\ , \tag{19}$$

can be interpreted as the density of probability of finding first particle at point $\bar{r}_1$, second - at point $\bar{r}_2$, etc. in time-moment t. Besides, the average value of quantum mechanical observable A is defined as follows

$$< A >= \int d^3r_1 \ldots d^3r_M\ \psi^*(\bar{r}_1, \ldots, \bar{r}_M, t)\ A\ \psi(\bar{r}_1, \ldots, \bar{r}_M, t)\ . \tag{20}$$

[5] $p_{iA} = -i\hbar\frac{\partial}{\partial x_{iA}}$.

4 Many-body quantum mechanics for twisted N-enlarged Newton-Hooke space-times

Let us now turn to the main aim of our investigations - to the quantum mechanical model of many particles defined on quantum space-times (9). In first step of our construction we extend the described in second section spaces to the whole algebra of momentum and position operators as follows

$$[\,\hat{x}_{1A},\hat{x}_{2B}\,] = if_{\kappa_a}(t)\delta_{AB} \quad , \quad [\,\hat{x}_{1A},\hat{x}_{3B}\,] = [\,\hat{x}_{2A},\hat{x}_{3B}\,] = [\,\hat{p}_{iA},\hat{p}_{jB}\,] = 0\;, \tag{21}$$

$$[\,\hat{x}_{iA},\hat{p}_{jB}\,] = i\hbar\delta_{ij}\delta_{AB} \;\;;\;\; i,j=1,2,3\;. \tag{22}$$

One can check that relations (21), (22) satisfy the Jacobi identity and for deformation parameters κ_a approaching zero become classical.
Next, by analogy to the commutative case (see formula (11)) we define the following multi-particle hamiltonian operator

$$H(\bar{\hat{p}}_1,\ldots,\bar{\hat{p}}_M;\bar{\hat{r}}_1,\ldots,\bar{\hat{r}}_M) = \sum_{A=1}^{M}\left(\frac{\bar{\hat{p}}_A^2}{2m_A} + V_A(\bar{\hat{r}}_A)\right) + \frac{1}{2}\sum_{A\neq B}V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)\;, \tag{23}$$

with $\bar{\hat{p}}_A = [\,\hat{x}_{1A},\hat{p}_{2A},\hat{p}_{3A}\,]$ and $\bar{\hat{r}}_A = [\,\hat{x}_{1A},\hat{x}_{2A},\hat{x}_{3A}\,]$.
In order to analyze the above system we represent the noncommutative operators $(\hat{x}_{iA},\hat{p}_{iA})$ by classical ones (x_{iA},p_{iA}) as (see e.g. [33], [41])

$$\hat{x}_{1A} = x_{1A} - \frac{1}{2\hbar}f_{\kappa_a}(t)p_{2A} \quad , \quad \hat{x}_{2A} = x_{2A} + \frac{1}{2\hbar}f_{\kappa_a}(t)p_{1A}\;, \tag{24}$$

$$\hat{x}_{3A} = x_{3A} \quad , \quad \hat{p}_{iA} = p_{iA}\;. \tag{25}$$

Then, the hamiltonian (23) takes the form

$$\begin{aligned}
&H(\bar{p}_1,\ldots,\bar{p}_M;\bar{r}_1,\ldots,\bar{r}_M,t) = \\
&= \sum_{A=1}^{M}\Bigg[\,\frac{\bar{p}_A^2}{2m_A} + V_A\left(\bar{\hat{r}}_A = \left(x_{1A} - \frac{1}{2\hbar}f_{\kappa_a}(t)p_{2A}, x_{2A} + \frac{1}{2\hbar}f_{\kappa_a}(t)p_{1A}, x_{3A}\right)\right) + \\
&\qquad + \frac{1}{2}\sum_{A\neq B}V_{AB}\left(\bar{\hat{r}}_A = \left(x_{1A} - \frac{1}{2\hbar}f_{\kappa_a}(t)p_{2A}, x_{2A} + \frac{1}{2\hbar}f_{\kappa_a}(t)p_{1A}, x_{3A}\right),\right. \\
&\quad \left. ,\quad \bar{\hat{r}}_B = \left(x_{1B} - \frac{1}{2\hbar}f_{\kappa_a}(t)p_{2B}, x_{2B} + \frac{1}{2\hbar}f_{\kappa_a}(t)p_{1B}, x_{3B}\right)\right)\Bigg]\;,
\end{aligned} \tag{26}$$

and, consequently, the corresponding Schroedinger equation in the position representation looks as follows

$$
\begin{aligned}
&i\frac{\partial}{\partial t}\psi(\bar{r}_1,\ldots,\bar{r}_M,t) = \\
= \;&\left\{ \sum_{A=1}^{M} \left[\frac{1}{2m_A}\Delta_A + V_A\left(\bar{\hat{r}}_A = \left(x_{1A} + \frac{i}{2}f_{\kappa_a}(t)\partial_{2A}, x_{2A} - \frac{i}{2}f_{\kappa_a}(t)\partial_{1A}, x_{3A}\right)\right) + \right.\right. \\
&\quad + \frac{1}{2}\sum_{A\neq B} V_{AB}\left(\bar{\hat{r}}_A = \left(x_{1A} + \frac{i}{2}f_{\kappa_a}(t)\partial_{2A}, x_{2A} - \frac{i}{2}f_{\kappa_a}(t)\partial_{1A}, x_{3A}\right),\right. \\
,\;\;&\left.\left.\left. \bar{\hat{r}}_B = \left(x_{1B} + \frac{i}{2}f_{\kappa_a}(t)\partial_{2B}, x_{2B} - \frac{i}{2}f_{\kappa_a}(t)\partial_{1B}, x_{3B}\right)\right)\right]\right\}\psi(\bar{r}_1,\ldots,\bar{r}_M,t)\;.
\end{aligned}
\tag{27}
$$

Further, we expand the hamiltonian function (27) in Taylor series up to the terms linear in deformation parameter κ_a, i.e. to the terms linear in function $f_{\kappa_a}(t)$; then, we have[6]

$$
\begin{aligned}
&H(\bar{p}_1,\ldots,\bar{p}_M;\bar{r}_1,\ldots,\bar{r}_M,t) = \\
= \;&\sum_{A=1}^{M}\left(\frac{\bar{p}_A^2}{2m_A} + V_A(\bar{r}_A)\right) + \frac{1}{2}\sum_{A\neq B} V_{AB}(\bar{r}_A,\bar{r}_B) + \\
&\quad + \left[\sum_{A=1}^{M}\left(-\frac{\partial V_A(\bar{\hat{r}}_A)}{\partial \hat{x}_{1A}}\cdot\frac{1}{2\hbar}p_{2A} + \frac{\partial V_A(\bar{\hat{r}}_A)}{\partial \hat{x}_{2A}}\cdot\frac{1}{2\hbar}p_{1A}\right) + \right. \\
+ \;&\frac{1}{2}\sum_{A\neq B}\left(-\frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{1A}}\cdot\frac{1}{2\hbar}p_{2A} + \frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{2A}}\cdot\frac{1}{2\hbar}p_{1A} + \right. \\
- \;&\left.\left.\left.\frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{1B}}\cdot\frac{1}{2\hbar}p_{2B} + \frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{2B}}\cdot\frac{1}{2\hbar}p_{1B}\right)\right]\right|_{f_{\kappa_a}(t)=0}\cdot f_{\kappa_a}(t) + \\
+ \;&\mathcal{O}(\kappa_a)\;,
\end{aligned}
\tag{28}
$$

with the corresponding wave equation given by

$$
\begin{aligned}
&i\frac{\partial}{\partial t}\psi(\bar{r}_1,\ldots,\bar{r}_M,t) = \\
= \;&\left\{\sum_{A=1}^{M}\left(\frac{1}{2m_A}\Delta_A + V_A(\bar{r}_A)\right) + \frac{1}{2}\sum_{A\neq B} V_{AB}(\bar{r}_A,\bar{r}_B) + \right. \\
&\quad + \left[\sum_{A=1}^{M}\left(\frac{\partial V_A(\bar{\hat{r}}_A)}{\partial \hat{x}_{1A}}\cdot\frac{i}{2}\partial_{2A} - \frac{\partial V_A(\bar{\hat{r}}_A)}{\partial \hat{x}_{2A}}\cdot\frac{i}{2}\partial_{1A}\right) + \right. \\
+ \;&\frac{1}{2}\sum_{A\neq B}\left(\frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{1A}}\cdot\frac{i}{2}\partial_{2A} - \frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{2A}}\cdot\frac{i}{2}\partial_{1A} + \right.
\end{aligned}
\tag{29}
$$

[6] We denote by $\mathcal{O}(\kappa_a)$ the higher order terms in deformation parameter κ_a.

$$+ \quad \frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{1B}} \cdot \frac{i}{2}\partial_{2B} - \frac{\partial V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)}{\partial \hat{x}_{2B}} \cdot \frac{i}{2}\partial_{1B} \Big) \Big] \Big|_{f_{\kappa_a}(t)=0} \cdot f_{\kappa_a}(t) +$$
$$+ \quad \mathcal{O}(\kappa_a) \; \} \, \psi(\bar{r}_1,\ldots,\bar{r}_M,t) \; .$$

Consequently, we see that space-time noncommutativity (6) generates in the hamiltonian (23) two types of additional dynamical terms. First of them arises from the single-particle potential $V_A(\bar{\hat{r}}_A)$ while the second one corresponds to the correlations $V_{AB}(\bar{\hat{r}}_A,\bar{\hat{r}}_B)$. Of course, for deformation parameters κ_a approaching zero all additional "potential" terms disappear.

Let us now turn to the mentioned in pervious section the system of M particles moving "in" and interacting "by" the Coulomb potential. Then, in accordance with formulas (28) and (29) the corresponding hamiltonian function as well as the corresponding Schroedinger equation take the form

$$\begin{aligned} H(\bar{p}_1,\ldots,\bar{p}_M;\bar{r}_1,\ldots,\bar{r}_M,t) &= \sum_{A=1}^{M}\left(\frac{\bar{p}_A^2}{2m_A} - \frac{Ze^2}{|\bar{r}_A|}\right) + \frac{1}{2}\sum_{A\neq B}\frac{e^2}{|\bar{r}_A-\bar{r}_B|} + \\ &\quad - \sum_{A=1}^{M}\frac{Ze^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}_A|^3}\cdot L_{3A} + \\ &\quad + \frac{1}{2}\sum_{A\neq B}\frac{e^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}_A-\bar{r}_B|^3}\cdot(L_{3B}+L_{3A}) + \\ &\quad - \frac{1}{2}\sum_{A\neq B}\frac{e^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}_A-\bar{r}_B|^3}\cdot(G_{AB}+G_{BA}) + \mathcal{O}(\kappa_a)\;, \end{aligned} \tag{30}$$

and

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(\bar{r}_1,\ldots,\bar{r}_M,t) &= \Bigg[\sum_{A=1}^{M}\left(\frac{1}{2m_A}\Delta_A - \frac{Ze^2}{|\bar{r}_A|}\right) + \frac{1}{2}\sum_{A\neq B}\frac{e^2}{|\bar{r}_A-\bar{r}_B|} + \\ &\quad - \sum_{A=1}^{M}\frac{Ze^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}_A|^3}\cdot L_{3A} + \\ &\quad + \frac{1}{2}\sum_{A\neq B}\frac{e^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}_A-\bar{r}_B|^3}\cdot(L_{3B}+L_{3A}) + \\ &\quad - \frac{1}{2}\sum_{A\neq B}\frac{e^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}_A-\bar{r}_B|^3}\cdot(G_{AB}+G_{BA}) + \mathcal{O}(\kappa_a)\Bigg] \times \\ &\quad \times \;\; \psi(\bar{r}_1,\ldots,\bar{r}_M,t)\;. \end{aligned} \tag{31}$$

respectively, with $L_{3A} = x_{1A}p_{2A} - x_{2A}p_{1A}$ and $G_{AB} = x_{1B}p_{2A} - x_{2B}p_{1A}$. Particulary, in the case of single particle, for canonical deformation $f_{\kappa_a}(t) = \kappa_a$ we reproduce the noncommutative model of hydrogen atom proposed in [30] and [31]

$$H(\bar{p},\bar{x}) \;=\; \frac{\bar{p}^2}{2m} - \frac{Ze^2}{|\bar{r}|} - \frac{Ze^2\kappa_a}{2\hbar|\bar{r}|^3}\cdot L_3 + \mathcal{O}(\kappa_a)\;, \tag{32}$$

while for more complicated (time-dependent) functions $f_{\kappa_a}(t)$, we get the one-particle system described by

$$H(\bar{p}, \bar{x}, t) \;=\; \frac{\bar{p}^2}{2m} - \frac{Ze^2}{|\bar{r}|} - \frac{Ze^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}|^3} \cdot L_3 + \mathcal{O}(\kappa_a) \; . \tag{33}$$

It is well-known, that the solution of the corresponding (associated with (33)) Schroedinger equation can be found with use of time-dependent perturbation theory [43]. It looks as follows

$$\psi(\bar{r}, t) = \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} c_{nlm}(t) \mathrm{e}^{iE_n(t-t_0)} \psi_{nlm}(\bar{x}) \; , \tag{34}$$

where symbols E_n and ψ_{nlm} denote eigenvalues and eigenfunctions for hydrogen atom, while coefficients $c_{nlm}(t)$ are defined as the solutions of the following differential equations

$$\begin{aligned} \frac{dc_{nlm}(t)}{dt} \;&=\; -\frac{1}{i\hbar} \sum_{n'=0}^{\infty} \sum_{l'=0}^{n-1} \sum_{m'=-l}^{l} \left(\psi_{nlm}(\bar{r}), \frac{Ze^2 f_{\kappa_a}(t)}{2\hbar|\bar{r}|^3} \cdot L_3 \psi_{n'l'm'}(\bar{r}) \right) c_{n'l'm'}(t_0) \; \cdot \\ &\cdot \quad \mathrm{e}^{i\omega_{nn'}(t-t_0)} \quad ; \quad \omega_{nn'} \;=\; \frac{1}{\hbar}(E_n - E_{n'}) \; . \end{aligned} \tag{35}$$

Hence, in accordance with prescription (15), the solution of multiparticle wave equation (31) with neglected correlation potential $V_{AB}(|\bar{r}_A - \bar{r}_B|)$ and vanishing $\mathcal{O}(\kappa_a)$-terms takes the form

$$\psi(\bar{r}_1, \ldots, \bar{r}_M, t) = \psi_1(\bar{r}_1, t) \cdots \psi_M(\bar{r}_M, t) \; , \tag{36}$$

with one-particle functions $\psi_A(\bar{r}_A, t)$ given by (34).

Finally, it should be noted that the average values of energy operators (27), (28) and (30) can be found with use of the formula (20).

5 Final remarks

In this article we construct the quantum model of M nonrelativistic particles moving in noncommutative space-time (9). The corresponding Schroedinger equation for arbitrary stationary potential is provided and, in particular, there is analyzed the distinguished example of such system - the set of M particles moving "in" and interacting "by" the Coulomb potential. It should be noted, however, that by analogy to the investigations performed in article [30], one can ask about more physical features (such as for example the energy spectrum or the Lamb shift) of the model defined by Hamiltonian (30). Besides, it should be added, that the presented considerations give a starting point for the construction of Dirac quantum mechanics for many particles defined on the relativistic counterpart of modified space-time (5). The studies in these directions already started and are in progress.

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References

[1] H.S. Snyder, Phys. Rev. 72, 68 (1947)

[2] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B 331, 39 (1994)

[3] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997); hep-th/9612084

[4] A. Connes, M.R. Douglas, A. Schwarz, JHEP 9802, 003 (1998); hep-th/9711162

[5] N. Seiberg and E. Witten, JHEP 9909, 032 (1999); hep-th/9908142

[6] S. Coleman, S.L. Glashow, Phys. Rev. D 59, 116008 (1999); hep-ph/9812418

[7] R.J. Protheore, H. Meyer, Phys. Lett. B 493, 1 (2000); astro-ph/0005349

[8] L.A. Takhtajan, *"Introduction to Quantum Groups"*; in Clausthal Proceedings, Quantum groups 3-28 (see High Energy Physics Index 29 (1991) No. 12256)

[9] V. Chari, A. Pressley, *"A Guide to Quantum Groups"*, Cambridge University Press, Cambridge, 1994

[10] S. Zakrzewski, *"Poisson Structures on the Poincare group"*; q-alg/9602001

[11] Y. Brihaye, E. Kowalczyk, P. Maslanka, *"Poisson-Lie structure on Galilei group"*; math/0006167

[12] M. Daszkiewicz, Mod. Phys. Lett. A 27 (2012) 1250083; arXiv: 1205.0319 [hep-th]

[13] R. Oeckl, J. Math. Phys. 40, 3588 (1999)

[14] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004); hep-th/0408069

[15] M. Daszkiewicz, Mod. Phys. Lett. A 23, 505 (2008); arXiv: 0801.1206 [hep-th]

[16] M. Chaichian, A. Demichev, P. Presnajder, Nucl. Phys. B 567, 360 (2000); hep-th/9812180

[17] M. Chaichian, A. Demichev, P. Presnajder, J. Math. Phys. 41, 1647 (2000); hep-th/9904132

[18] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264, 331 (1991)

[19] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57 (1992)

[20] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116 (2006); hep-th/0508083

[21] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, Comm. Math. Phys. 150, 495 (1992)

[22] P. Aschieri, L. Castellani, A.M. Scarfone, Eur. Phys. J. C 7, 159 (1999); q-alg/9709032

[23] A. Deriglazov, JHEP 0303, 021 (2003); hep-th/0211105

[24] A.E.F. Djemaï, *"On noncommutative classical mechanics"*; hep-th/0309034

[25] J.M. Romero and J.D. Vergara, Mod. Phys. Lett. A 18, 1673 (2003); hep-th/0303064

[26] J.M. Romero, J.A. Santiago, J.D. Vergara, Phys. Lett. A 310, 9 (2003); hep-th/0211165

[27] Y. Miao, X. Wang, S. Yu, *"Classical mechanics on noncommutative space with Lie-algebraic structure"*; arXiv: 0911.5227 [math-ph]

[28] E. Harikumar, A.K. Kapoor, *"Newton equation on the kappa space-time and the Kepler problem"*; arXiv: 1003.4603 [hep-th]

[29] M. Daszkiewicz, C.J. Walczyk, Phys. Rev. D 77, 105008 (2008); 0802.3575 [mat-ph], [hep-th]

[30] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001); hep-th/0010175

[31] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Euro Phys. J. C 36, 251 (2004); hep-th/0212259

[32] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, Nucl. Phys. B 11, 383 (2001)

[33] A. Kijanka, P. Kosinski, Phys. Rev. D 70, 127702 (2004); hep-th/0407246

[34] M. Daszkiewicz, C.J. Walczyk, Acta Phys. Pol. B 40, 293 (2009); arXiv: 0812.1264 [hep-th]

[35] G. Amelino-Camelia, Phys. Lett. B 510, 255 (2001); hep-th/0012238

[36] G. Amelino-Camelia, Mod. Phys. Lett. A 17, 899 (2002); gr-qc/0204051

[37] M. Maggiore, Phys. Rev. D 49, 5182 (1994); hep-th/9305163

[38] S.M. Rama, Phys. Lett. B 519, 103 (2001); hep-th/0107255

[39] R.J. Szabo, Phys. Rept. 378, 207 (2003); hep-th/0109162

[40] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Ann. Phys. 243, 90 (1995); hep-th/9312153

[41] J. Lukierski, P. Stichel, W.J. Zakrzewski, Ann. Phys. 260, 224 (1997); hep-th/9612017

[42] J.D. Anderson, P.A. Laing, E.L. Lau, A.S. Lin, M.M. Nieto, S.G. Turyshev, Phys. Rev. Lett. 81, 2858 (1998); gr-qc/9808081

[43] L.J. Schiff, *Quantum Mechanics*, McGraw-Hill, New York 1968

[44] S. Kryszewski, *Quantum Mechanics*, University of Gdansk, Gdansk 2010

[45] G. Fiore, J. Wess, Phys. Rev. D 75, 105022 (2007); hep-th/0701078

[46] M. Daszkiewicz, C.J. Walczyk, Mod. Phys. Lett. A 26, 819 (2011); arXiv:1104.3243 [hep-th]

[47] M. Chaichian, P.P. Kulish, A. Tureanu, R.B. Zhang, Xiao Zhang, J. Math. Phys. 49: 042302 (2008); arXiv:0711.0371 [hep-th]